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ASYMMETRICAL VARIABLE DEMAND MULTI-MODE TRAFFIC EQUILIBRIUM PROBLEMS: EXISTENCE AND UNIQUENESS OF SOLUTIONS AND .. OLUTION ALGORITHM

by

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ABSTRACT

The purpose of this paper is to explore the theoretical properties of the variable demand multi-modal traffic equilibrium models proposed in [8]. When the link cost user cost functions are monotone (increasing) and the mode demand functions are strictly monotone (decreasing), the solution of these models is unique in the origin/destination demands and travel times. The existence of a solution is demonstrated by constructing an upper semi-continuous mapping whose fixed points are traffic equilibria. Existence then follows from Kakutani's fixed point theorem [10]. A solution algorithm that is more general than that proposed in [8] is discussed.

The purpose of this paper is to provide conditions for the existence and uniqueness of solution to variable demand multi-modal traffic equilibrium models proposed in [8]. Since, in general, these models are not equivalent to solving corresponding optimization problems [5], [6], [14], existence of a solution necessitates the demonstration of the existence of a fixed point of an appropriately chosen mapping. In a recent paper, Dafermos [7] demonstrates existence and uniqueness of the fixed demand multi-modal (multi-class) traffic equilibrium model and constructs a solution algorithm which is a contraction mapping, under the assumption that the user cost functions are uniformly monotone (increasing) with the flows. The uniqueness proof provided in this paper, for the more general variable demand problem, requires weaker monotonicity conditions of the user cost functions and the existence proof requires that the demand functions be strictly monotone (decreasing) functions of the origin/destination travel times. The existence proof is demonstrated by the application of Kakutani's fixed point theorem [9] in a way similar to the approach of Braess and Koch [4]. In work reported several years ago, Netter [13], [14] claims that Netter and Senders [12] provided an existence proof for the variable demand multi-model traffic equilibrium problem by an application of Brouwer's fixed point theorem; I had no access to reference [12], however, by the description in [13], that approach sums different.

The paper is organized in the following way: Section 1 presents the notation used and the variable demand

multi-modal traffic equilibrium model; section 2 recalls some of the properties of single mode variable demand problems, existence of a solution, uniqueness of the solution and the properties of a slightly more general algorithm that the one proposed in [8] are presented in sections 3, 4 and 5 respectively; section 6 contains a discussion of the practical implications of these results and a comparison with the method proposed by Dafermos [7].

1. NOTATION AND MODEL FORMULATION

Consider a network of nodes n, $n \in N$, and directed arcs a, $a \in A$. The nodes represent origins, destinations or intersections of arcs. The arcs represent the streets of an urban area, the highways of a regional area, etc. Each arc is accessible to flow of several modes of traffic, designated by index i. The individual user cost for travel by mode m on arc a, $\bar{s}_a^m$ depends on the flows v_a^m of all the modes, i = 1,...,M. That is

$$\bar{s}_a^m = s_a^m(v_a^1, v_a^2, \ldots, v_a^M) \quad , \text{ all a}$$

The vector cost function $s_a(v)$ is composed of the individual user cost functions for all the modes m for arc a, where v is the vector $(v_a^1, v_a^2, \ldots, v_a^M)$. The user cost, functions are supposed to be monotone, i.e.

$$s_a^m(v) \leq s_a^m(\hat{v}) \quad \text{when } v \leq \hat{v} \ ,$$

positive and continuous.

The origin/destination demands $g_{pq,m}$ are given by a set of demand functions $G_{pq,m}$ which depend on the travel costs u^m_{pq} by all modes m between origins p and destinations q

$$g_{pq,m} = G_{pq,m}(u^1_{pq}, u^2_{pq}, \ldots, u^M_{pq}) \quad \text{all } (p,q)$$

The vector demand function $G_{pq}(u_{pq})$ is composed of the demand functions for all the modes m for pair (p,q), where u_{pq} is the vector $(u^1_{pq}, u^1_{pq}, \ldots, u^M_{pq})$. The restriction of the demand functions with respect to all modes except one, say mode m, is obtained by keeping all the vector u_{pq} fixed except for u^m_{pq} and is denoted $\bar{G}_{pq,m}$. It is assumed that the restrictions $\bar{G}_{pq,m}$ are non negative, strictly decreasing continuous functions of u^m_{pq} and that there is an upper bound for each of these functions; that is, for the smallest possible u^m_{pq}, denoted $\underline{u}^m_{pq}$, the corresponding implied demand $\bar{G}_{pq,m}(\underline{u}^m_{pq})$ is less than or equal to that upper bound.

On the network, the demands $g_{pq,m}$ subdivide into path flows $h^m_{k,pq}$, where k is a path index. The path flows are feasible ($h^m_{k,pq} \in F$) if

$$\sum_k h^m_{k,pq} = G_{pq,m}(u_{pq}) \quad \text{all } (p,q),m$$

and (1)

$$h^m_{k,pq} \geq 0 \quad \text{all } k,(p,q),m$$

The corresponding arc flows v_a are given by

$$v^m_a = \sum_k \sum_{(p,q)} \delta^m_{ak,pq} \cdot h^m_{k,pq} \qquad (2)$$

and $$v_a = \sum_m v^m_a$$

where $\delta_{ak,pq}^{m}$ is an indicator function which identifies the arcs belonging to path $(_{k,pq}^{m})$.

For simplicity of the exposition, any flow, be it a path flow or an arc flow, that satisfies (1) and (2) will be denoted feasible, and the space of feasible demands and flows denoted F. The multimodal variable demand traffic equilibrium model is based on Wardrop's [16] 'user-optimized' behavioral principle which states that travel costs on all routes from any origin to any destination are equal, while all unused routes have equal or greater travel costs. The model is constructed by supposing that the above condition holds for all modes. Let $\bar{s}_{k,pq}^{m} = \sum_{a} \delta_{ak,pq}^{m} . h_{k,pq}^{m}$. The model is

$$\bar{s}_{k,pq}^{m} = u_{pq}^{m} \quad \text{if } h_{k,pq}^{m} > 0$$

$$\bar{s}_{k,pq}^{m} = u_{pq}^{m} \quad \text{if } h_{k,pq}^{m} = 0 \qquad \text{, all } (p,q), m \qquad (3)$$

and the flows $h_{k,pq}^{m}$ feasible, $h_{k,pq}^{m} \in F$.
(The notation used is different than that of [4], [7] or [13], however is consistent with that used in [8].

In the next section, the single mode variable demand traffic equilibrium model is reviewed and the uniqueness of the solutions is characterized.

2. THE SINGLE MODE VARIABLE DEMAND MODEL

Consider the model obtained by fixing all the flows and demands except that of mode m. Let $\tilde{s}_a^m(v_a^m)$ be the restriction of the user cost functions for mode m; these are positive, monotone, continuous and differentiable functions. Solving this restricted problem is equivalent [2] to obtaining the minimum of

$$Z = \sum_a \int_0^{v_a^m} \tilde{s}_a^m(x)dx - \sum_{(p,q)} \int_0^{g_{pq}^m} \tilde{G}_{pq,m}^{-1}(y)dy \qquad (4)$$

subject to feasibility conditions, i.e. $(v_a^m, g_{pq,m}) \in F^m$. Since Z is a convex function, as a consequence of the monotonicity of $\tilde{s}_a^m(v_a^m)$ and the strict monotonicity of the inverse demand function $\tilde{G}_{pq,m}^{-1}$, the optimal value of the objective function, Z^*, is unique. Its value may be calculated by adapting any algorithm that minimizes a convex function subject to linear constraints, such as described in [15], for instance. The values of the arc flows v_a^m are not necessarily unique. To ascertain this consider the solution vector (v^m, g^m) where $v^m = (v_a^m)$ and $g^m = (g_{pq}^m)$ for a chosen ordering of the arcs a and the origin destination pairs (p,q). It is possible to characterize the solution of (4) by the following development.

Proposition 1. (v^{*m}, g^{*m}) is a solution to the variable demand traffic equilibrium problem if and only if for feasible (v^m, g^m)

$$\tilde{s}^m(v^{*m})(v^m - v^{*m}) - \tilde{G}_m^{-1}(g^{*m})(g^m - g^{*m}) \geq 0 \qquad (5)$$

Demonstration. It is known, see for instance [11], that $f(x)$, convex and differentiable, is minimized on the convex set Ω at x^* if and only if the directional derivative $\nabla f(x^*)(y - x^*) \geq 0$ for all $y \in \Omega$. (5) is merely an application of this result for the objective function (4)□

It is possible now to determine the consequences of the monotonicity assumptions on the user cost functions on the unicity the solution vector.

Proposition 2. (g^{*m}) are unique, however (v^{*m}) are not necessarily unique.

Demonstration. The strict monotonicity of $\tilde{G}_m^{-1}$ implies that $[\tilde{G}_m^{-1}(g^m) - \tilde{G}_m^{-1}(\hat{g}^m)](g^m - \hat{g}^m) < 0$, when $g^m \neq \hat{g}^m$. The monotonicity of $\tilde{s}_a^m(v_a^m)$ implies that for $v_a^m \neq \hat{v}_a^m$ $[\tilde{s}_a^m(v_a^m) - \tilde{s}_a^m(\hat{v}_a^m)](v_a^m - \hat{v}_a^m) \geq 0$. The result follows by supposing that there exist two distinct optimal solutions (v^{*m}, g^{*m}) and $(v^{\#m}, g^{\#m})$. Writing the condition (5) results in

$$\tilde{s}^m(v^{*m})(v^{\#m} - v^{*m}) - \tilde{G}_m^{-1}(g^{*m})(g^{\#m} - g^{*m}) \geq 0 \qquad (6)$$

and

$$\tilde{s}^m(v^{\#m})(v^{*m} - v^{\#m}) - \tilde{G}_m^{-1}(g^{\#m})(g^{*m} - g^{\#m}) \geq 0 \qquad (7)$$

By adding (6) and (7) and imposing the monotonicity condition of $\tilde{s}_a$ and $\tilde{G}_m^{-1}$ the result follows □

The interpretation of proposition 2 is quite simple. For instance, where all the user cost functions are constant, the solution is trivially obtained by computing shortest paths between all origin destination pairs and

assigning the corresponding demand to the arcs on the shortest paths. Since the shortest path is not necessarily unique, the non uniqueness of the arc flows is evident.

Consider now the algorithm that solves (4) and provides, given a feasible (v^m,g^m), an optimal solution (v^{*m},g^{*m}). It may be considered to be a point to set mapping which sends each feasible (v,g) into a subset of the feasible space of flows and demands F^m. Thus, it may be considered to be a mapping A^m:F → all subsets of F_m. The following proposition (see also Lemma 1 in [4]) is of central importance for showing the existence of the solution for the multi-modal model.

Proposition 3. The mapping A^m is upper semicontinuous (u.s.c.).

Demonstration. The proof given here is different than the proof of Lemma 1 in [4]. It uses the following characterization of u.s.c. mappings [3]: a mapping B: X → Y is u.s.c. if Bx is compact for all x∈X and for every open set G in Y, the set B^+G is open. The mapping A^m depends on the flows and demands for all other modes (v^i,g^i), $i \neq m$. Since A^m provides the set of optimal solutions of a convex program, it is well known that this set is convex and bounded, hence compact. The verification of the second condition is not as easy as the first. Let G_ε be all the points that are at a distance $\varepsilon > 0$ of some element (v^m,g^m) of A^m, for a given $(\bar{v}^i,\bar{g}^i)$, $i \neq m$; then there must exist a $\gamma > 0$ such that if $(\bar{\bar{v}}^i,\bar{\bar{g}}^i)$, $i \neq m$ is perturbed, i.e.

$|\bar{v}^i - \bar{\bar{v}}^i| < \gamma$, $i \neq m$ and $|\bar{g}^i - \bar{\bar{g}}^i| < \gamma$, then $A^m(\bar{\bar{v}}^i, \bar{\bar{g}}^{=i}) i \neq m \in G_\varepsilon$. Each perturbation of $(\bar{v}^i, \bar{g}^i)$ results in a translation of the user cost functions or the demand functions for mode m. Since these functions are supposed to be continuous and the minimum of the function (4) is unique, it follows that this minimum varies continuously with changes in $(\bar{v}^i, \bar{g}^i)$ as all the slopes of the functions in (4) are locally bounded. Hence the second condition is verified as well □

In the next section the existence of the solution for the multi-modal model is demonstrated.

3. EXISTENCE OF SOLUTIONS FOR THE MULTI-MODAL MODEL

The method used in this section is similar to that in [4] however the details differ somewhat. Form the composite map obtained by the cartesian product $A = \prod_{i=1}^{M} A^i$ of the M u.s.c. mappings that correspond to the restrictions of the multi-mode problem for each mode i. It is known that [3] the map A is u.s.c. in the cartesian product of the target spaces, which is the feasible set of multi-modal demands and flows $(v,g) \in F$. Hence A: F→F, where each $(v^i, g^i) \in A^i(v,g)$ is a restricted user-optimized equilibrium with respect to its own mode. It follows that

Proposition 4. There is at least one solution of the multi-modal variable demand traffic equilibrium model.

Demonstration. The result follows by applying Kakutani's [10] fixed point theorem which states that an u.s.c. set-valued

mapping from a compact set X to the closed convex sets in X has at least one fixed point. The space of the vectors {v,g} is compact since the flows satisfy nonnegativity and conservation of flow equations and the demands are nonnegative and bounded □

It is possible now to state a characterization of the solutions to the multi-modal model. Since each (v^{*i}, g^{*i}) is a restricted user optimized equilibrium with respect to its own mode i, the result of proposition 1 holds for every i. Therefore we have the following

Corollary. (v^*, g^*) is a solution of the multi-mode model if and only if

$$s(v^*)(v - v^*) - \tilde{G}^{-1}(g^*)(g - g^*) \geq 0 \qquad (8)$$

where $s(v)$ is the vector of user cost functions for all modes and $\tilde{G}^{-1}(g)$ is the vector of the inverses of the restricted demand functions for all the modes.

Dafermos [7] atributes a similar result for the fixed demand problem to B. T. Smith. I had no access to Smith's work.

It remains now to characterize the uniqueness of the solutions, a most important consideration for application.

4. UNIQUENESS OF SOLUTIONS FOR THE MULTI-MODAL MODEL

By using the corollary of the previous section it is straightforward to prove the unicity property of the solutions, in a manner similar to proposition 2.

Proposition 5. Let the user cost functions $s(v)$ be monotone (increasing) and the inverse of the demand functions $G(u)$ be strictly monotone (decreasing). Then g^* is unique but v^* is not necessarily unique.

Demonstration. Analogous to the proof of proposition 2 and by using the fact that $[\tilde{G}^{-1}(g^*) - \tilde{G}^{-1}(g)](g^* - g) < 0$ is equivalent to $(u^* - u)[\tilde{G}(u^*) - \tilde{G}(u)] < 0$ □

The implications of proposition 5 are that unique origin/destination demands, and hence travel costs, may be obtained under weaker conditions on the cost functions than those required by Dafermos [7]. However, the Jacobian of the mapping $-G(u)$ must be positive definite, which implies that the main dependence of $G_m(u)$ should be on u_m, a condition which may not always be satisfied.

It remains to state a solution algorithm, which is a repetitive application of the mappings A^m and discuss some of its properties.

5. A SOLUTION ALGORITHM AND ITS CONVERGENCE

Suppose that the model admits a unique solution. The following algorithm, which is slightly more general than the one that I proposed in [8] for a two-mode problem

may be used to obtain this solution.

Step 0. Select an initial feasible solution $(v,g)^0 \in F$; set $k = 1$.

Step 1. Compute $(v^{*i},g^{*i})^k \in A^i(v,g)^{k-1}$, $i = 1,\ldots,M$ (that is, solve M single mode variable demand problems).

Step 2. If $|g^k - g^{k-1}| > \varepsilon$, where ε is a suitable tolerance set $k \leftarrow k+1$ and return to step 1, otherwise terminate.

The properties of the algorithm is the subject of

Proposition 6. If the algorithm outlined above terminates, it obtains the unique solution of the multi-modal problem.

Demonstration: If the algorithm terminates, then it obtains the solution to the multi-modal problem, since otherwise the test in step 2 would have failed □

It is interesting to note that the mapping A is u.s.c. and hence it is closed [3]. However this is not sufficient to prove the convergence of the algorithm, since it is necessary to show that a sequence $\{v,g\}^k$ converges to the fixed point $\{v,g\}^*$. Empirically, the convergence observed with practical applications of the algorithm proposed in [8] and reported in [9] is rapid: at most 3 executions of step 1 above were required.

It is interesting to make some remarks on the

specialization of the above algorithm to the problem considered in reference [8], that of combining a modal split function with road and transit assignment models. There are two modes, but only one distinct demand function, since the transit demand $g_{pq}^{tr} = g_{pq}^{tot} - g_{pq}^{au}$, where 'tr' denotes transit, 'tot' denotes total trips and 'au' denotes automobile. Thus, solving the variable demand equilibrium problem for the auto mode, implicitly determines the transit demand. Therefore, a necessary condition for the uniqueness of the solution of that model is that the modal choice function be strictly monotone decreasing for all feasible values of u_{pq}^{au}, u_{pq}^{tr}. This may not be trivial to ascertain in particular applications.

6. DISCUSSION

The implication of these results for practical applications is considerable. If the algorithm may be shown to converge and the model admits a unique solution, then solving the multi-mode variable demand traffic equilibrium problem is no more difficult than solving the single mode problem. In particular, if the adaptation of the Frank and Wolfe algorithm is used to solve this problem, see [15] for instance, then the method becomes a sequence of shortest path calculations and one dimensional convex function minimifations. In contrast, the algorithm proposed by Dafermos [7] for the fixed demand version of this problem requires the computation of an upper bound of the maximum eigenvalue of a symmetric positive definite

matrix of the same dimension as the number of arcs. In medium size applications, this number may be of the order of 3000, making this task difficult, at the very least. On the other hand, the approach adopted in [7] provides a proof of the convergence of the algorithm and the rate of convergence

Recently, Abdulaal and Leblanc [1] proposed an algorithm for solving the two mode (modal split-assignment) problem which is intimately related to the one that proposed in [8]. A convergence proof for that algorithm is still lacking as well I am also aware of research carried out by H. Aashtiani under Tom Magnanti's direction, based on casting the multi-modal problem in the form of a nonlinear complementarity problem. I had no access, to date, to the technical details of this work.

Last, but not least, I would like to point out that the single mode variable demand traffic equilibrium model, where the demand functions depend on travel costs to all destinations, from a given origin, may also be analysed by the same approach and it is the subject of a forthcoming paper.

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